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A deduction of the Landau-Lifshitz equation of motion without mass renormalization in classical electrodynamics

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Abstract : A new expression for the large distance radiated power of a spinless charged point particle is obtained As a consequence of this and by using the balance of energy criterion, the Landau-Lifshitz equation of motion in classical electrodynamics is deduced Neither the mass renormalization process nor the Lorentz-Dirac equation is required In order to show the goodness of the Landau-Lifshitz equation of motion, the case of a pulse is analyzed The new expression predicts that the large distance radiated power is increased

Keywords Reaction radiation force, Larmor formula, charged particle

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1. Introduction

By using a simple approximation of the Lorentz-Dirac equation [1] [LD] for point charged particles in classical electrodynamics. Landau and Lifshitz [2] obtained an equation which has been mathematically supported by Spohn [3]. Rohrlich [4] physically reinforced the goodness of the Landau-Lifshitz equation [LL] noticing that it is a second order differential equation which does not present runaway solutions or preaccelerations. It is very important to note that we can obtain the LL equation by substituting the acceleration and its derivatives in the LD reaction term by the value of the acceleration from the Lorentz equation [L] Then, the LL is considered as a first order approximation in τ_0 of the LD equation. In fact, its derivation consists of the observation that τ_0 is so small that it needs to be taken into account only up to first order. Indeed, according to Rohrlich [5]

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and Spohn [3], the LD equation must be restricted to its critical surface yielding the LL equation and consequently, the last one represents the correct equation of motion for a spinless classical point charge Moreover, Ares de Parga *et al* [6] have proposed a physical deduction of the LL equation of motion which implies a change in the concept of the radiated power Indeed, in the non-relativistic case its large distance radiated power coincides with the O'Connell's [7] proposal of the radiated power Since the LD equation presents a lack of energy balance, not due to the covariant form, but in the sense of not respecting the law of inertia (it allows a self-force in the absence of an external force [5]) the LL equation seems to be the better physical option to describe the motion of a spinless point charged particle Moreover, for special situations, the LL solutions are completely different from the LD solutions as there are the cases of runaway solutions and preaccelerations Therefore, LL equation cannot be considered as a simple first order aproximation in τ_0 of the LD equation On the other hand, all the deductions of the LL equation are based on using the LD equation as a point of departure In the rigorous process of obtaining the LD equation, the self-energy, an infinite term, is ignored This is called the mass renormalization Nevertheless, this is not rigorous as Synge [8] has noticed Indeed, Synge [8] tried to solve the arbitrariness of ignoring the self-energy (the infinite term) by proposing an exponential decreasing factor in the fields due to the point particle in order to eliminate any kind of divergences The objective of this paper consists of proposing the LL equation of motion as the exact equation which describes the motion of a spinless point charged particle The deduction is based neither by requiring a mass renormalization nor by using the LD equation Moreover, the present article claims a different expression for the large distance radiated power The essential idea consists of proposing that the radiation emitted by a point charge is due exclusively to the external forces driving its motion This will justify why LL equation of motion doesn't lead to notorious pathologies as runaway solutions and preaccelerations

The paper is organized as follows in Section 2 we deduce a new expression for the large distance radiated power of a spinless point charged particle By using some kinematic identities, we obtain the LL equation of motion and also some equivalent representations of the LL equation are proposed In order to show the goodness of the LL equation a typical pulse case is analyzed in Section 3 The new expression obtained here for the large distance radiated power will bring an interesting result Finally, some concluding remarks are given in Section 4

2. The Landau-Lifshitz equation and the new expression for the radiated power

Let us propose that the motion of a spinless point charged particle is governed by the following equation (Gaussian units and signature − 2 are used)

$$ma^{\mu} = \frac{q}{c}F^{\mu\nu}v_{\nu} + f^{\mu}_{rad}, \tag{1}$$

where q, c, $F^{\mu\nu}$ and f^{μ}_{rad} represent the charge, the speed of light, the electromagnetic field-strength tensor and an unknown reaction force Since the reaction force is unknown, it is impossible to define the electromagnetic field and consequently also the electromagnetic field-strength tensor. We need to suppose that for $(q/cm)F^{\mu\nu}v_\nu$ weakly enough, the particle will follow the L equation. Consequently, we can experimentally prove the Maxwell equations but we can define the relativistic Maxwell stress tensor and compute the radiated power emmitted by the particle just when the particle follows the L equation. Even if we don't know the motion of the particle in general, we can always suppose that if we restrict the particle to move following L equation, the radiated power will coincide with the regular relativistic Larmor formula. Therefore, we need to add to the applied electromagnetic field another unknown force G^μ, perhaps of electromagnetic origin too, such that the motion of the charge is governed by the L force : that is :

$$f^\mu_{rad} + G^\mu = 0\,, \tag{2}$$

then

$$ma^\mu = \frac{q}{c}F^{\mu\nu}v_\nu + f^\mu_{rad} + G^\mu = \frac{q}{c}F^{\mu\nu}v_\nu \tag{3}$$

First of all, we know that the large distance radiation rate of energy-momentum emitted by a spinless point charged particle which motion is driven by the L force, is expressed by [11] :

$$\frac{d}{d\tau}P^\mu = \frac{\tau_0 m}{c^2}a^2 v^\mu, \tag{4}$$

where $\tau_0 = \left(2q^2/3mc^3\right)$ represents the characteristic time of the charged particle and $a^2 = a^\mu a_\mu$. The situation given by eqs. (2) and (3) represents a "gedanken" experiment in the sense that perhaps it is not possible to assert the physical existence of such a force during an arbitrary interval of time. Since the motion of the particle is L-like, we can always express the energy-momentum rate emitted by the particle as :

$$\frac{d}{d\tau}P^\mu_L = -\frac{\tau_0 m}{c^2}a^2 v^\mu = -\tau_0\frac{q^2}{c^2 m}F^{\alpha\beta}v_\beta F_{\alpha\gamma}v^\gamma v^\mu = -\tau_0\frac{q^2}{c^2 m}F^2 v^\mu, \tag{5}$$

where we have introduced the notation F^2 as $F^{\alpha\beta}v_\beta F_{\alpha\gamma}v^\gamma = F^{\alpha\beta}v_\beta F_{\alpha\gamma}v^\gamma$. The Anzatz now consists of considering that when G^μ is not anymore applied, the expression of the large distance radiated power (energy-momentum) emitted by the particle is still described by

$$\frac{d}{d\tau}P^\mu = -\tau_0\frac{q^2}{c^4 m}F^{\alpha\beta}v_\beta F_{\alpha\gamma}v^\gamma v^\mu = -\tau_0\frac{q^2}{c^2 m}F^2 v^\mu. \tag{6}$$

If we want to find an equation of motion of a charge, the large distance radiated power emitted by the particle must be a part of it. Thus, we can finally conclude that the equation of motion can be written as

$$ma^{\mu} = \frac{q}{c}F^{\mu\nu}v_{\nu} + f^{\mu}_{rad} = \frac{q}{c}F^{\mu\nu}v_{\nu} + D^{\mu} + \tau_0 \frac{q^2}{c^4 m}F^2 v^{\mu}, \tag{7}$$

where an extra term D^{μ} appears in order to accomplish the balance of energy when eq (7) is contracted with v_{μ} Let us now investigate as what is the value of D^{μ} Therefore D^{μ} must satisfy

$$D^{\mu}v_{\mu} + \tau_0 \frac{q^2}{c^4 m}F^2 v^{\mu}v_{\mu} = D^{\mu}v_{\mu} + \tau_0 \frac{q^2}{c^4 m}F^2 = 0 \tag{8}$$

There are many possibilities of finding D^{μ} But, once we have a good candidate such that eq (8) is satisfied, we can always add a vector H^{μ} with $H^{\mu}v_{\mu}$ Nevertheless, by following Dirac's hypothesis of simplicity and, as we will see at the end of this section by the principle of correspondence, we propose the following solution

$$D^{\mu} = \tau_0 \frac{q}{c}\frac{dF^{\mu\nu}v_{\nu}}{d\tau}\bigg|_{L-path} = \tau_0 \frac{q}{c}\frac{dF^{\mu\nu}v_{\nu}}{d\tau_L}, \tag{9}$$

where we have introduced a new variable called the Lorentz proper time or L time, τ_L The meaning of this time is the following one in each point along the real path of the particle, we consider the L path that possesses the same 4-velocity of the particle Then the derivative with respect to the L time represents the derivative with respect to the proper time along the L path (see Figure 1) That is the derivative $(d/d\tau)|_{L\ path}$ is represented by $d/(d\tau_L)$ Consequently,

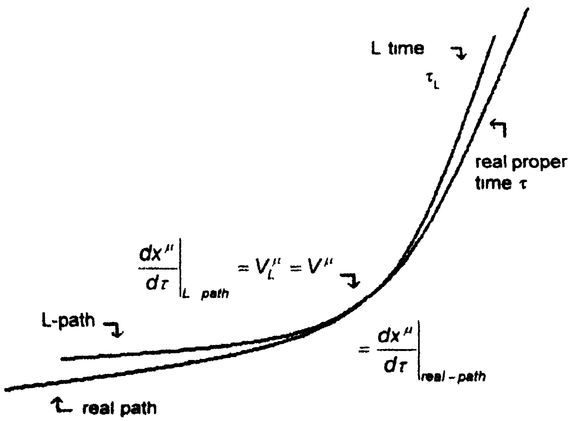

Figure 1. L time

$$\frac{dv^\mu}{d\tau}\Big|_{L-path} \quad \frac{dv^\mu}{d\tau_L} = \frac{q}{cm}F^{\mu\nu}v_\nu \tag{10}$$

and for the field-strength tensor,

$$\frac{d}{d\tau}F^{\mu\nu}\Big|_{L-path} \quad \frac{d}{d\tau_L}F^{\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial x^\alpha}\frac{dx^\alpha}{d\tau_L} = \frac{\partial F^{\mu\nu}}{\partial x^\alpha}v^\alpha, \tag{11}$$

since the 4-vector velocity coincides for the L or real path of the particle Therefore, eq (8) is satisfied since

$$\tau_0\frac{q}{c}\frac{dF^{\mu\nu}v_\nu}{d\tau_L}v_\mu + \tau_0\frac{q}{c^2m}F^2$$

$$= \tau_0\frac{q}{c}\frac{\partial F^{\mu\nu}}{\partial x^\alpha}v^\alpha v_\nu v_\mu + \tau_0\frac{q}{c}F^{\mu\nu}\frac{q}{cm}F_{\nu\alpha}v^\alpha v_\mu + \tau_0\frac{q^2}{c^2m}F^2 = 0, \tag{12}$$

where the antisymmetry of the field-strength tensor have been used Eq (12) shows that the choice of the L-path permits to satisfy eq (8) This means that the equation of motion may be expressed as

$$ma^\mu = \frac{q}{c}F^{\mu\nu}v_\nu + \tau_0\frac{q}{c}\frac{dF^{\mu\nu}v_\nu}{d\tau_L} + \frac{q^2}{c^4m}F^2v^\mu \tag{13}$$

If we develop the second term in the right hand side of eq (13) by considering the identities of eqs (10) and (11), and further by using the antisymmetry of $F^{\mu\nu}$, we arrive at

$$ma^\mu = \frac{q}{c}F^{\mu\nu}v_\nu + \tau_0\left|\frac{q}{c}\frac{\partial F^{\mu\nu}}{\partial x^\alpha}v_\nu v^\alpha - \frac{q^2}{c^4m}F^{\mu\nu}F_{\nu\alpha}v^\alpha + \frac{q^2}{c^4m}F^2v^\mu\right|, \tag{14}$$

which represents the LL equation of motion

Some important conclusions can be inferred from eqs. (13) and (14)

(A) Eq (13) and eq. (14) are equivalent ; that is eq. (13) is another representation of the LL equation and, as Rohrlich [4,5] has noticed, the LL equation is a second order equation which does neither permit runaway solutions nor preaccelerations

(B) The reaction term, written as

$\frac{q}{c}\frac{dF^{\mu\nu}v_\nu}{d\tau_L} + \frac{q^2}{c^4m}F^2v$, possesses two terms, one of which is a quasi-reversible reaction term and another which represents the large distance radiated power which is not reversible. The term quasi-reversible is due to the derivative with respect to the Lorentz time.

(C) Following the same reasoning, it is straightforward to obtain an equivalent representation of the LL equation of motion by starting from the Eliezer equation [14,6]

(D) It can be thought that by using the Larmor formula whose deduction is free of divergences [11], the mass renormalization is avoided because we do not directly calculate the interaction of the charge with its own radiation field. Indeed, instead of directly calculating the interaction of the charge with its own radiation field which will give a divergence, or by introducing an exponential factor following Synge [8] or by combining retarded and advanced potentials as Unruh [10] did, or by using theories similar to the perfect absorber of Wheeler and Feynman [13], we prefer to evaluate the radiated power by using an indirect method which avoids the divergence. In this form, we are reinforcing the idea that the reaction force is due exclusively to the external forces driving its motion On the other hand, it requires an arbitrary assumption, since the D^{μ} vector, introduced *ad hoc* to achieve the energy balance, is not fully determined by its inner product with the proper velocity. Nevertheless, as we noticed before, the choice of D^{μ} has been done by using the Dirac's hypothesis of simplicity [1]. Moreover, the arbitrariness can be established by noticing that Eliezer [14] and later on Ford and O'Connell [15] have found a non relativistic equation, based on quantum arguments, which coincides with the non relativistic version of the LL equation. Even if Eliezer [14] dealt with point charges and Ford and O'Connell [15] developed their theory for structured charged particles, they found the same result for the reaction force, $\tau_0\left(d\boldsymbol{F}/dt\right)$. Therefore, the principle of correspondence justified our choice of D^{μ}. Consequently the LL equation of motion represents the correct equation of motion for a point charged particle.

(F) The form of the reaction term in the Landau-Lifshitz equation shows that the essential idea of the theory is that the radiation emitted by the point charge is due exclusively to the external forces driving its motion and the self-force due to its own fields does not contribute to the radiation.

3. The pulse case

In order to understand the physical aspects of the LL equation and our new theory of the radiated power, let us analyze a typical pulse. We will just consider an electric pulse and will neglect the magnetic part of the pulse that has to be carried out with it. Let us consider that the particle is at rest at a time $\tau = -\infty$ and the electric pulse is represented by :

$$\boldsymbol{E} = Ef\left(\tau\right)\bar{i}. \tag{15}$$

The LL equation can be expressed as :

$$c\ddot{t} = w\left[f(\tau) + \tau_0\dot{f}(\tau)\right]\dot{x} \tag{16}$$

and

$$x = w\left[f(\tau) + \tau_0 \dot{f}(\tau)\right] ct$$

where $w = qE/mc$ Putting $ct = c\cosh\Omega$ and $x = c\sinh\Omega$, we obtain

$$\dot{\Omega} = w\left[f(\tau) + \tau_0 f(\tau)\right] \tag{17}$$

Therefore,

$$\Omega = w\int^{\tau} f(\tau)d\tau + \tau_0 w\left[f(\tau) - f(-\infty)\right] \tag{18}$$

In the case when the electric field is a constant, that is $f(\tau) = 0$, and by setting the initial conditions of the particle at rest at $\tau = 0$, the result is

$$\Omega = wf\tau\,, \tag{19}$$

as in the case of the L force This result has already been notice before by some authors [17,16]

For a pulse described by Gaussian function (see Figure 2)

$$f(\tau) = \frac{(1\,\text{sec})}{\sqrt{\pi}\,\alpha}\exp-\frac{\tau^2}{\alpha^2}\,, \tag{20}$$

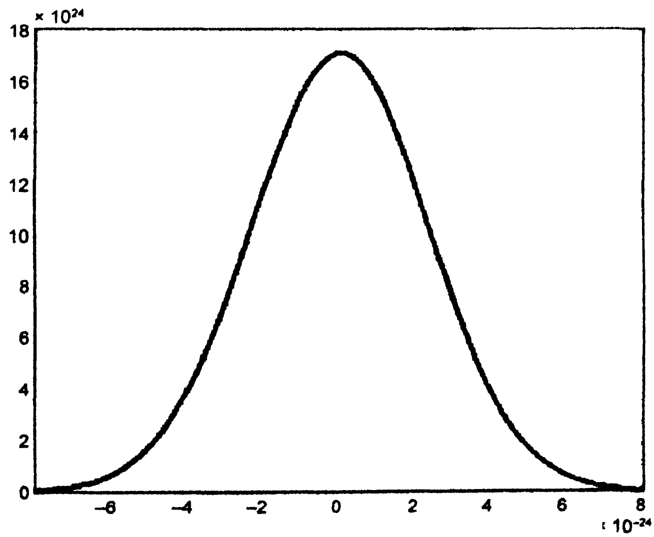

Figure 2. The Gaussian function $f(\tau) = \frac{(1\,\text{sec})}{\sqrt{\pi}\,\alpha}\exp-\frac{\tau^2}{\alpha^2}$ with $\alpha = 3\ 3 \times 10^{24}$s (sec = s)

where α represents a parameter in seconds that if we limit it to 0, $f(\tau)$ transforms into a delta function, that is : $\lim_{\alpha\to 0} f = \delta(\tau)$ We obtain, following eq. (18),

$$\Omega = \frac{w}{2}\left(1 + erf\,\frac{\tau}{\alpha}\right) + \tau_0 w \frac{1}{\sqrt{\pi}\,\alpha} \exp - \frac{\tau^2}{\alpha^2} . \tag{21}$$

The final results are (see Figures 3, 4, 5 and 6)

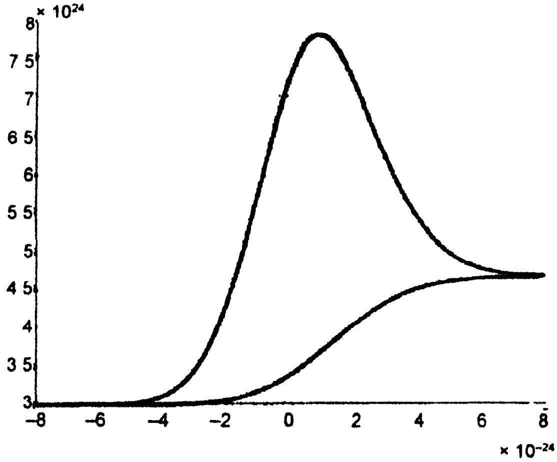

Figure 3. ct with $w = \frac{qE}{mc} = 1$, that is E = 5 7 × 10^{-8} stavolt cm^{-1}

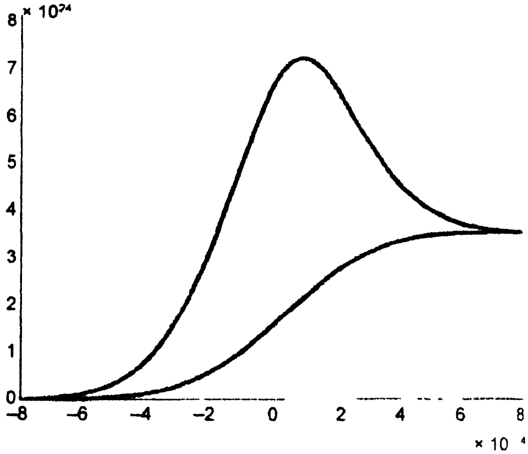

Figure 4. x with $w = \frac{qE}{mc} = 1$, that is τ_0 6 26 $\times 10^{-24}$ s The lower curve corresponds to the L motion

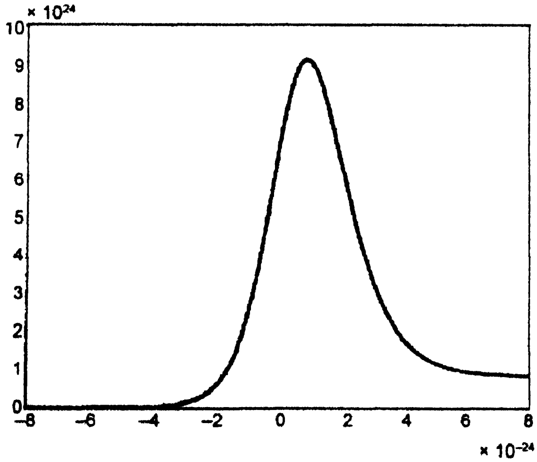

Figure 5. ct with $w = \frac{qE}{mc} = 4$, and $\tau_0 = 6\ 26 \times 10^{-24}$

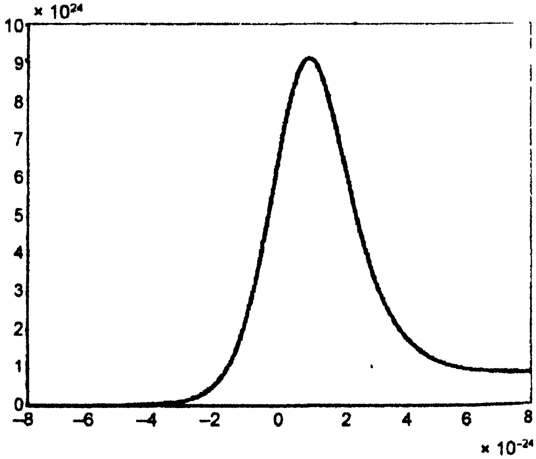

Figure 6. x with $w = \frac{qE}{mc} = 4$ and $\tau_0 = 6\ 26$ $\times 10^{-24}$ s

$$\dot{c}t = c\cosh w\left[\frac{1}{2}\left(1+erf\frac{\tau}{\alpha}\right)+\tau_0\frac{1}{\sqrt{\pi}\alpha}\exp-\frac{\tau^2}{\alpha^2}\right] \tag{22}$$

and
$$\dot{x} = c\sinh w\left[\frac{1}{2}\left(1+erf\frac{\tau}{\alpha}\right)+\tau_0\frac{1}{\sqrt{\pi}\alpha}\exp-\frac{\tau^2}{\alpha^2}\right].$$

Looking at the figures it is easy to notice that there is a particular behavior of the solutions of the LL equation. Indeed, the particle increases the velocity until a maximum and then it slows down. That is : the particle suffers a deacceleration (negative acceleration) during the pulse and then the motion tends to be similar to the one predicted by the L equation (see Figures 3 and 4). If we are interested in analyzing the consequences of these phenomena, namely, negative accelerations, related with the large distance radiated power, we need to calculate

$$A = -\tau_0\frac{q^2}{c^2m}F^2\nu^0 = \tau_0 m w^2 f^2 c\cosh w\left[\frac{1}{2}\left(1+erf\frac{\tau}{\alpha}\right)+\tau_0\frac{1}{\sqrt{\pi}\alpha}\exp-\frac{\tau^2}{\alpha^2}\right]\cdot \tag{23}$$

Before the particle deaccelerates, it obtains more velocity for the LL equation than for the L equation, and consequently during such a period the LL particle radiates more energy than the L particle (we are considering the same expression, eq. (5), for calculating the radiated powers for both the LL and the L equations). Moreover, the effect grows when the strength is increased (see Figure 5, 6, 7 and 8).

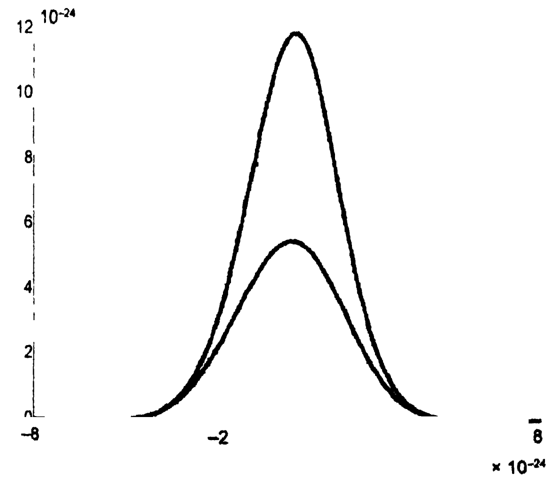

Figure 7. Large distance radiated power for w = 1 with $\tau_0 = 6.26\times10^{24}$s. The lower curve corresponds to the L motion.

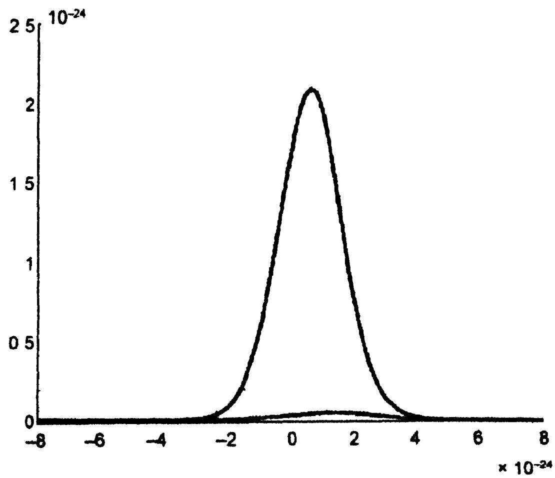

Figure 8. Large distance radiated power for w = 4 with $\tau_0 = 6.26\times10^{24}$s. The lower curve corresponds to the L motion

4. Concluding remarks

First of all, it has to be pointed out that in the deduction of the LL equation no mass renormalization has been used. A new expression for the large distance radiated power has been obtained. An analysis of the new reaction term has been done and by an interpolation with the LD theory the attached and large distance reaction terms have been exposed. It has been noticed that by using quantum reasons. Eliezer [14], Ford and O'Connell [15] got an equation which coincides with the nonrelativistic LL equation Consequently, we can propose that the LL equation represents the correct equation of motion of a point or structured charged particle. That is : the new expression for the radiated power and the LL equation of motion represent exact results and not approximations.

It is important to notice that the essential result is that the radiation emitted by a point charge is due exclusively to the external forces driving the motion and that the self-force due to its own fields does not contribute to the radiation even though it may affect the motion of the charge. The only way to prove our proposal is an experiment. Nevertheless, it is very difficult to design an experiment to test the motion of a charge. But, by measuring the drift of the center of motion of a spinless point charge for a constant magnetic field [19], it will be possible to check the validity of the LL equation.

We have to point out that the regular definition of the Poynting vector has to be changed in order that it still represents the flux of energy. Same thing happens with the stress tensor.

Due to the fact that we consider the proposal as an exact solution, we can analyze situations where the radiation terms are of the same order of magnitude of the applied force terms. As a consequence of this, a typical example, the pulse, has been solved showing the advantages of using both the LL equation and the new expression for the large distance radiated power instead of using a mixed theory as in the Baylis proposal [18]. For the pulse case, an interesting maximum of the velocity appears when the LL equation is considered instead of L equation. The charge accelerates and deaccelerates in such a manner that when the pulse is finished, the motion is identical to the one that will present by considering the L equation. The final velocities coincide for both cases and in order to accomplish this the radiation is higher for the LL solutions than for the L ones.

Finally, even if the phenomenon of negative acceleration is bigger for high fields (see Figure 4, w = 1, and Figure 6, w = 4), it has to be noticed that it is more important when high frequency pulses are considered. Indeed, we took a Gaussian pulse with α = 6.24 × 10^{-24} sec. It has to be pointed out that the pulses and the energies considered in the paper represent points in the field *vs* energy diagram that belong to the Shen's Zone [20], that is, the physical domain where classical electrodynamics with radiation reaction may be used to describe the motion of a charge.

Acknowledgment

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